Bhavesh Study Circle

AMTI (NMTC) - 2004

RAMANUJAN CONTEST - INTER LEVEL

1. When $1^{2003} + 2^{2003} + 3^{3003} + ... + 2003^{2003}$ is divided by 2004, then the remainder is

(A) 0 (B) 1 (C) 1002 (D) 2003

2. Each number from 1, 2, 3, ..., 100 (decimal scale) is written in base 6 and their product is also written in base 6. Then the number of zeroes at the end of this product is

(A) 24 (B) 48 (C) 18 (D) 97

3. Given that $a_1 a_2$, ..., a_{2004} are distinct positive real numbers then $\frac{a_1}{a_2} + \frac{a_2}{a_3} + ... + \frac{a_{2003}}{a_{2004}} + \frac{a_{2004}}{a_1}$ is

(A) less than 2004 (B) less than 1

(C) greater than 2004 (D) equal to 1

4. Two chords of a circle bisect each other. Then

(A) The chords are of equal length but less than the length of the diameter.

(B) The chords are of unequal length.

(C) The chords should be perpendicular bisectors of each other.

(D) The chords are any two diameters.

5. A man tosses a fair coin till he gets a head. The probability that he gets head in at most n tosses is

(A) $\frac{1}{n}$ (B) $\frac{1}{2^n}$ (C) $1 - \frac{1}{n}$ (D) $1 - \frac{1}{2^n}$

6. - - - -

(A) - - - (B) - - - (C) - - - (D) - - -

7. - - - -

(A) - - - (B) - - - (C) - - - (D) - - -

8. - - - -

(A) - - - (B) - - - (C) - - - (D) - - -

9. - - - -

(A) - - - (B) - - - (C) - - - (D) - - -

10. - - - -

(A) - - - (B) - - - (C) - - - (D) - - -

11. Kumar never lies except on Tuesdays. On Tuesdays, he always lies. On how many days of the week can he say "If I did not lie yesterday then I will lie tomorrow."

(A) 1 (B) 2 (C) 3 (D) 4

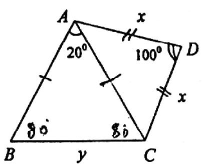
12. The range of function $f(r) = {}_{7-r}P_{r-3}$ where ${}_mP_n$ represents permutations of 'm' things taken 'n' at a time and where r is a non-negative integer is

(A) {1, 2, 3, 4, 5} (B) {1, 2, 3, 4} (C) {1, 2} (D) {1, 2, 3}

13. The roots of $64x^3 - 144x^2 + 92x - 15 = 0$ are in A.P. Then the difference between the largest and smallest is

(A) $\frac{1}{2}$ (B) $\frac{3}{4}$ (C) $\frac{7}{8}$ (D) 1

14. In the following figure, if AD = DC = x and BC = y then AB = AC = ?



(A) x + y (B) 2y + x (C) 2y (D) 2x

15. ABCD is a tetrahedron. The number of planes from which the distances to A, B, C, D are equal is

(A) 0 (B) 5 (C) 4 (D) 3

16. The three last digits of 7^{9999} are

(A) 263 (B) 143 (C) 343 (D) 523

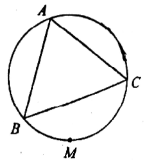
17. Each of the faces of a cube is coloured by a different colour. How many of the colourings are distinct ?

(A) 6 (B) 30 (C) 18 (D) 24

18. Which of the following is a continuous function f satisfying 3f(2x + 1) = f(x) + 5x ?

(A) f(x) = 2x + 5 (B) f(x) = x + 1 (C) $f(x) = x - \frac{3}{2}$ (D) $f(x) = x + \frac{1}{2}$

19. A, B, C, M are four points on a circle such that ABC forms an equilateral triangle and M lies on the arc BC as in the figure. Which of the following holds good ?



(A) |MA| + |MB| + |MC| = 2|AB| (B) |MA| = |MB| + |MC|

(C) $|MA|^2 = |MB|^2 + |MC|^2$ (D) $|MA|^2 + |MB|^2 + |MC|^2$ = Area of ΔABC

20. The positive numbers x and y satisfy xy = 1. The minimum value of $\frac{1}{x^4}+\frac{1}{4y^4}$ is

(A) $\frac{1}{2}$ (B) $\frac{5}{8}$ (C) 1 (D) $\frac{5}{4}$

21. A cubic polynomial P is such that P(1) = 1, P(2) = 2, P(3) = 3 and P(4) = 5. The value of P(6) is

(A) 7 (B) 10 (C) 13 (D) 16

22. In ΔABC, the altitude from A to BC meets BC at D and the altitude from B to CA meets AD at H. If AD = 4, BD = 3, CD = 2 then the length of HD is

(A) $\frac{\sqrt{5}}{2}$ (B) $\frac{3}{2}$ (C) $\sqrt{5}$ (D) $\frac{5}{2}$

23. A fair coin is tossed 10,000 times. The probability p of obtaining at least 3 heads in a row satisfies

(A) $0 \le p \le \frac{1}{4}$ (B) $\frac{1}{4} \le p \le \frac{1}{2}$ (C) $\frac{1}{2} \le p < \frac{3}{4}$ (D) $\frac{3}{4} \le p < 1$

24. The number of different positive integer triplets (x, y, z) satisfying the equations $x^2 + y - z = 100$ and $x + y^2 - x = 124$ is

(A) 0 (B) 1 (C) 2 (D) 3

25. If $\phi(n)$ denotes the number of positive integers less than n and prime to n and A = {n|ϕ(n) = 17} then the number of elements in A is

(A) infinite (B) 16 (C) 17^2 (D) zero

26. For what values of n is an n–digit number uniquely determined from the sum and product of its digits ?

(A) 1 (B) 2 (C) 3 (D) 4

27. The number of polynomials P(x) satisfying the equation $P(x^2) + 2x^2 + 10x = 2xP(x + 1) + 3$ is

(A) 2 (B) 1 (C) 3 (D) infinite

28. Let n be a positive integer. Then the number of common factors of $n^2 + 3n + 1$ and $n^2 + 4n + 3$ is

(A) (n + 1) (B) n – 1 (C) 2 (D) 1

29. If f(n + 1) = f(n) + n for all $n \geq 0$ or f(0) = 1 then f(200) equals

(A) 21100 (B) 21000 (C) 20900 (D) 19900

30. The rays PX and PY cut off arc AB and CD of a circle with radius 4. If the length of arc CD is 2 times the length of the arc AB and length of CD is $\frac{4\pi}{5}$, then the angle APB is

(A) 9^0 (B) 10^0 (C) 12^0 (D) 18^0

PART - A

1. ABCD is a rhombus in which B is obtuse. Perpendiculars are drawn from B to the sides and 'a' is the length of each perpendicular. The distance between their feet is 'b'. Area of the rhombus is

(A) $\frac{a^4}{b\sqrt{4a^2-b^2}}$ (B) $\frac{a^4}{2b\sqrt{4a^2-b^2}}$ (C) $\frac{2a^4}{b\sqrt{4a^2-b^2}}$ (D) $\frac{2a^4}{b\sqrt{4b^2-a^2}}$

2. Let n be a natural number. The number of n's for which $(n^4 + 2n^3 + 2n^2 + 2n + 1)$ is a perfect square is

(A) 1 (B) 2 (C) 7 (D) 0

3. [x] denotes the greatest integer not exceeding x. If $a = 2 + \sqrt{3}$ then the value of $a^n + a^{-n} - [a^n]$ for any positive integer n is

(A) 2 (B) $\sqrt{3}$ (C) 1 (D) 0

4. The quadrilateral ABCD is inscribed in a circle. The diagonals AC and BD cut at Q. The sides DA and CB are produced to meet at P. If CD = CP = DQ then the measure of $\angle CAD$ is

(A) 45^0 (B) 70^0 (C) 60^0 (D) 55^0

5. The expression $(4n^3 + 6n^2 + 4n + 1)$ is

(A) Composite for all natural numbers n.

(B) Prime for exactly two natural numbers n.

(C) Prime for infinitely many natural numbers n.

(D) Composite for odd n and prime for even n.

6. $a_1, a_2, \ldots$ is a sequence for which $a_1 = 2$, $a_2 = 3$ and $a_n = \frac{a_{n-1}}{a_{n-2}}$ for every natural number $n \geq 3$. The value of a_{2011} is

(A) 3 (B) $\frac{3}{2}$ (C) $\frac{2}{3}$ (D) 2

7. If 49 in base a and 94 in base b represent the same number in base 10, then the least possible value of this number is (in base 10)

(A) a square number (B) a four digit number necessarily

(C) a three digit square free number (D) a three digit prime

8. a, b are positive integers. If $21ab^2$ and $15ab$ are perfect squares, the minimum value of a + b is

(A) 56 (B) 65 (C) 23 (D) 42

9. ABC is a triangle. AB = 6 cm, BC = 9 cm, CA = 7 cm. Circles are drawn with centres at A, B, C. The circles with centres A, B touch externally. The circle with centre C touches these two circles internally. The sum of the radii of these circles (in cm) is

(A) 22 (B) 17 (C) 44 (D) 111

10. ABCD is a convex quadrilateral in which $AD = \sqrt{3}$, $\angle A = 60^0$, $\angle D = 120^0$ and AB + CD = 2AD. M is the midpoint of BC. Then DM =

(A) $\frac{\sqrt{3}}{2}$ (B) $\frac{3}{2}$ (C) $\frac{2}{3}$ (D) $\frac{2}{\sqrt{3}}$

11. A triangle has integer sides. One side is three times a second side. The length of the third side is 15. The greatest possible perimeter of the triangle is

(A) 43 (B) 46 (C) 48 (D) 52

12. If $\frac{x}{3y} = \frac{y}{2x-5y} = \frac{6x-15y}{x}$ and the expression $(-4x^2 + 36y - 8)$ takes the maximum value for x = m and y = n then m + n =

(A) 2011 (B) 2 (C) 3 (D) 17

13. ABC is a triangle. Perpendiculars BM and CN are drawn to the tangent at A to the circum circle of ΔABC. The tangent at A meets BC produced at D. If BC = 5 cm, the shorter perpendicular CN = 6 cm and AD = $5\sqrt{6}$ cm, then the area of the trapezoid BCNM (in cm^2) is

(A) 30 (B) 40 (C) 35 (D) 45

14. The number of integers λ for which the equation $x^3 - 3x + \lambda = 0$ has integer roots is

(A) 1 (B) 2 (C) 7 (D) infinite

15. A straight line passing through A(–1, –25) and parallel to the line 5x – 4y + 95 = 0 cuts the x and y axes at B and C respectively. Consider the unit grid in the plane. The number of squares in the grid containing the line segment BC in their interior is

(A) 0 (B) 15 (C) 27 (D) 5

PART - B

1. The function f(x) satisfies the condition $(x - 2) f(x) + 2f\left(\frac{1}{x}\right) = 2$ for all x ≠ 0. Then the value of f(2) is ______.
2. Let $s(x) = 6x^5 + 5x^4 + 4x^3 + 3x^2 + 2x + 1$. The zeros of f(x) are a, b, c, d, e. Then the value of (1 + a) (1 + b) (1 + c) (1 + d) (1 + e) = ______.
3. A quadrilateral inscribed in the circle has side lengths $\sqrt{20}, \sqrt{99}, \sqrt{22}$ and $\sqrt{97}$ in order. Taking $\pi = \frac{22}{7}$, the area of the circle is _______.
4. The number of rectangles that can be obtained by joining four of the 14 vertices of a 14 sided regular polygon is _______.
5. Let $a_0 = 1$, $a_{n+1} = 5a_n + 1$ for n ≥ 1. The remainder when a_{2011} is divided by 13 is ____.
6. ABCD is a rhombus of side 2 units. ∠B = 30^0. Then the area of the region within the rhombus such that every point in this region is closer to vertex B than to vertices A, C and D is ______.
7. In a rectangle ABCD where AB = 6, BC = 3, point P is chosen on AB such that ∠APD = 2∠CPB. Then AP = _______.
8. The perimeter of the octagon formed by the roots of the polynomial equation $Z^8 - 256 = 0$, plotted in the complex plane in order is ____.
9. Deleted.
10. If $f(x) = \frac{3x}{5x+4}$ and f(g(x)) = x, then the expression for the function g(x) is ______.
11. ABC is a triangle. P is any point inside the triangle. d_1, d_2, d_3 are the lengths of the perpendiculars to the sides BC, CA, AB respectively from P. h1, h_2, h_3 ar the altitudes to these sides respectively. The numerical value of $\frac{d_1}{h_1} + \frac{d_2}{h_2} + \frac{d_3}{h_3}$ is _____.
12. The equal sides of an isosceles triangle are each equal to 2011 cm. The length of the third side which makes the area of the triangle maximum is ______.
13. Both the roots of $x^2 - 63x + k = 0$ are prime numbers. The sum of the digits of k is ____.
14. The number of ordered pairs of integers (x, y) satisfying $x + y = x^2 - xy + y^2$ is _____.
15. ABC is a triangle in which C = 900. D lies on the segment BC such that BD = AC $\sqrt{6}$. E lies on the segment AC such that AE = CD $\sqrt{6}$. The acute angle between the lines AD and BE is _____.

PART - A

1. The sum of the squares of all real numbers satisfying the equation $x^{256} - 256^{32} = 0$ is
   (A) 8 (B) 128 (C) 512 (D) 65536

2. Two adjacent vertices of a square are on a circle of radius R and the other two vertices lie on a tangent to the circle. The length of the side of the square is
   (A) $\frac{4R}{3}$ (B) $\frac{6R}{5}$ (C) $\frac{8R}{5}$ (D) $\frac{3R}{2}$

3. The polynomial $x^{2n} + 1 (x + 1)^{2n}$ is not divisible by $(x^2 + x + 1)$ if n is equal to
   (A) 17 (B) 20 (C) 21 (D) 64

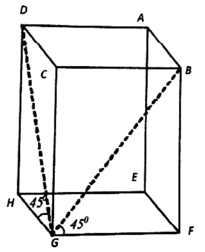
4. The remainder when 3^{302} is divided by 11 is
   (A) 10 (B) 9 (C) 8 (D) 7

5. If a, b and d are the lengths of a side, a shortest diagonal and a longest diagonal of a regular nonagon, then
   (A) $d^2 = a^2 + ab + b^2$ (B) $d^2 = a^2 + b^2$
   (C) $d = a + b$ (D) $b^2 = ad$

6. If a positive integer, after adding 100, becomes a perfect square, and also after adding 168, becomes another perfect square. (168 is added to the original positive integer), then the sum of the digits of the integer is
   (A) 12 (B) 11 (C) 8 (D) 7

7. ABC is a triangle inscribed in a circle. AD is the altitude of the triangle. DP is drawn parallel to AB to cut the tangent at A at P. Then ∠CPA is
   (A) Equal to 90^0 when triangle ABC is acute angled
   (B) Equal to 90^0 for any non-right angled triangle ABC
   (C) Greater than 90^0 for the angle A obtuse
   (D) Smaller than 90^0 for an acute angled triangle ABC

8. In the adjoining figure of a rectangular solid, it is given that $\angle DGH = 45^0$ and $\angle BGF = 60^0$. Then $\cos(\angle BGD)$ =



   (A) $\frac{\sqrt{6}-\sqrt{2}}{4}$ (B) $\frac{\sqrt{2}}{6}$ (C) $\frac{\sqrt{6}}{4}$ (D) $\frac{\sqrt{3}}{6}$

9. For the inequality $(1.25)^{1-x} < (0.64)^{2(1+\sqrt{x})}$.
   (A) There is no real value of x exists
   (B) There are infinitely many negative value of x exist
   (C) Any real value of x greater than 25 satisfies the inequation
   (D) There is exactly one positive and one negative integer only satisfying the equation

10. The number $(5^6 - 10^4)$ is
   (A) Negative (B) divisible by 10 (C) divisible by 100 (D) divisible by 9

11. There are 'a' points on a line and 'b' points on a parallel line. The number of triangles formed by these (a + b) points as vertices is
   (A) $\frac{ab(a+b-2)}{2}$ (B) $\frac{ab(a+b-1)}{2}$ (C) $\frac{ab(a+b-4)}{2}$ (D) $\frac{ab(a+b)}{2}$

12. The equation $x^4 + 16x - 12 = 0$ has
(A) all the roots real
(B) all the roots complex
(C) two roots real and two roots complex
(D) all the roots integers

13. A positive integer n has 60 divisors and 7n has 80 divisors. What is the greatest value of k such that 7^k divides n.
(A) 1 (B) 2 (C) 3 (D) 0

14. In the XY plane two points A(2, 2) and B(7, 7) are taken. R is the region in the first quadrant which consists of points C such that triangle ABC is an acute angled triangle. The closest integer to the area of the region R is
(A) 25 (B) 39 (C) 51 (D) 60

15. How many three digit numbers have distinct digits such that one digit is the average of the other two ?
(A) 104 (B) 112 (C) 256 (D) 12

PART - B

1. Let $f(x) = x^2 + bx + c$ where b, c are integers. If f(x) is a factor of both $x^4 + 6x^2 + 25$ and $3x^4 + 4x^2 + 28x + 5$, then the value of f(1) is _______.

2. Mahadevan's age is 'a' years, which is also the sum of the ages of his three children. His age b years ago was twice the sum of their ages. Then $\left(\frac{a}{b}\right)$ = _______.

3. $\angle$RST is an angle in the minor segment of a circle of centre O, then the angle (in degrees) RST less the angle ORT is _______.

4. A, B, C are three towns connected by straight roads from A to B, B to C and C to A. AB = 5 km, BC = 6 km, CA = 7 km. Two cyclists start simultaneoulsy form A and go in different roads with same speed. They meet at D. then BD = _______.

5. f is a linear function given by f(x) = ax + b and $f^{-1}(x) = bx + a$ when a, b are real. The value of a + b is equal to ________.

6. ABC is triangle. A point O is taken inside the triangle such that $\angle BOC = 120^0$. OD, OE, OF are drawn perpendiculars to the sides BC, CA, AB respectively. Then $\angle$EDF + EAF = ______.

7. ABC is an isosceles triangle in which AB = BC. BC is produced to D such that $\angle CAD = \frac{1}{2}\angle BAC$. If L is the foot of the $\perp$r from C to AD, then the value of $\frac{AL}{AD}$ = ____.

8. The values of x satisfying the inequality $x - \sqrt{1-|x|} < 0$ lie in $\left[-1, \frac{a}{2}\right]$. Then te value of a is ______.

9. For some real numbers a and b, the equation $8x^3 + 4ax^2 + 2bx + a = 0$ has three distinct positive roots. If the sum of the logarithms to base 2 of the roots is 5, the value of a is _____.

10. ABCD is a quadrilateral in the first equation of the coordinate axes. A is (3, 9), B is (1, 1), C is (5, 3) and D is (a, b). The quadrilateral formed by joining the mid-points of AB, BC, CD and DA is a square. The sum of the coordinates of the point D is ______.

11. The number of distinct four-tuples of numbers (a, b, c, d) of rational numbers satisfying $a\log_{10}2 + b\log_{10}3 + c\log_{10}5 + d\log_{10}7 = 2012$ is ______.

12. If the graph of $f(x) = ||x - 2| - a| - 3$ has exactly three x – intercepts, then a is equal to ____.

13. If we add the square of the digit in the tens place of a positive two digit number to the product of the digits of the number, we get 52. If we add the square of the digit in the units place of the number to the same product of the digits, we get 117. The two digit number is _______.

PART - A

1. The number of non–zero integers x such that $-10 < x < 15$ for which $3x^3 + 7x^2$ is the square of an integer is
(A) 5 (B) 6 (C) 0 (D) 1

2. The vertices of a triangle ABC are lattice points (that is, points with integer coordinates). Two of its sides have lengths which belong to the set $\{\sqrt{19}, \sqrt{2013}, \sqrt{2014}\}$. The maximum possible area of the triangle is
(A) 2013 (B) 1007 (C) 2014 (D) $\sqrt{2014}$

3. The minimum value of the terms of the sequence $\sqrt{\frac{7}{6}}+\sqrt{\frac{96}{7}}, \sqrt{\frac{8}{6}}+\sqrt{\frac{96}{8}}, \sqrt{\frac{9}{6}}+\sqrt{\frac{96}{9}}, \ldots\ldots, \sqrt{\frac{95}{6}}+\sqrt{\frac{96}{95}}$ is
(A) 6 (B) 7 (C) 8 (D) 4

4. ABCD is a cyclic quadrilateral in a circle of radius r. AB is a diameter of the circle. CD is parallel to AB. CD = b, AD = BC = a. The value of $\frac{2r^2 - a^2}{br}$ is
(A) 1 (B) 2 (C) 3 (D) 4

5. Positive integers a, b, c are chosen so that $a < b < c$ and the system of equations $2x + y = 2013$ and $y = |x - a| + |x - b| + |x - c|$ have exactly one solution. Then the minimum value of c is
(A) 760 (B) 1007 (C) 2013 (D) 2012

6. ABCD is a rectangle. P and Q are points on AB and BC respectively such that the area of triangle APD = 5, area of triangle PBQ = 4 and area of triangle QCD = 3, all areas in square units. Then the area of the triangle DPQ is square units is
(A) 12 (B) $\frac{20}{3}$ (C) $2\sqrt{21}$ (D) $\sqrt{21}$

7. The number of right triangles of integer length sides and the product of the leg lengths is equal to three times the perimeter is
(A) 0 (B) 1 (C) 2 (D) 3

8. Let S be the set of ordered triples (x, y, z) of real numbers for which $\log_{10}(x + y) = z$ and $\log_{10}(x^2 + y^2) = z + 1$. a, b are real numbers such that for all ordered triples (x, y, z) in S, we have $x^3 + y^3 = a \cdot 10^{3z} + b \cdot 10^{2z}$. Then the value of (a + b) is
(A) $\frac{15}{2}$ (B) $\frac{29}{2}$ (C) 15 (D) $\frac{39}{2}$

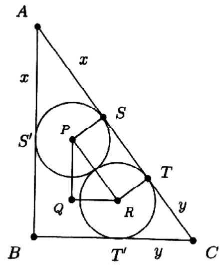
9. The internal bisector of $\angle A$ of a triangle ABC meets BC at P and $b = 2c$. If $9AP^2 + 2a^2 = k \cdot c^2$, then k is
(A) 8 (B) 3 (C) 19 (D) 18

10. The product of 5 odd primes is a five digit number of the form ab0ab where a, b are digits and the hundreds digit is zero. The number of such numbers is
(A) 0 (B) 9 (C) 13 (D) 18

11. The number of perfect square divisors of the product 1! 2! 3! 4! ..... 9! is
(A) 583 (B) 615 (C) 627 (D) 672

12. Srilekha, Priyanka, Vidya and Vishwa bought a gift for their classmate's birthday. The gift is with someone of the four. Each makes a statement. One statement is false and the other three are true.
Srilekha : I do not have the gift and Vidya does not have the gift.
Priyanka : I do not have the gift and Srilekha does not have the gift.
Vidya : I do not have the gift and Priyanka does not have the gift.
Vishwa : I do not have the gift and Srilekha does not have the gift.
The gift is with
(A) Srilekha (B) Priyanka (C) Vidya (D) Vishwa

13. In the figure below the triangle is a 3 – 4 – 5 sides triangle. Two equal circles are placed as in the figure. The radius of each circle is



(A) $\frac{4}{5}$ (B) $\frac{5}{8}$ (C) $\frac{6}{11}$ (D) $\frac{5}{7}$

14. The number of positive integer pairs (m, n) where n is odd, that satisfy $\frac{1}{m}+\frac{4}{n}=\frac{1}{12}$ is

(A) 1 (B) 2 (C) 3 (D) 11

15. The number of points common to the two graphs whose equations are $||x|-|y||+|x|+|y|=2$ and $2y=|2x-1|-3$ is

(A) 1 (B) 2 (C) 3 (D) 4

PART - B

16. When simplified, the value of $\frac{(10^4+324)(22^4+324)(34^4+324)(46^4+324)(58^4+324)}{(4^4+324)(16^4+324)(28^4+324)(40^4+324)(52^4+324)}$ is ______.

17. The least positive integer a such that the equation $\cos^2\pi(a-x) - 2\cos\pi(a-x) + \cos\frac{3\pi x}{2a}\cdot\cos\left(\frac{\pi x}{2a}+\frac{\pi}{3}\right)+2=0$ has a real root is ______.

18. The quadrilateral ABCD is inscribed in a circle. The diagonals AC and BD intersect at Q. The sides DA and CB are produced to meet at P. Given the CD = CP = DQ, the value of $\angle CAD$ = ______.

19. Dividing a three digit number by the number obtained from it by swapping its first and last digits we get 3 as the quotient and the sum of the digits of the original number as the remainder. The number of such 3-digit numbers is ______.

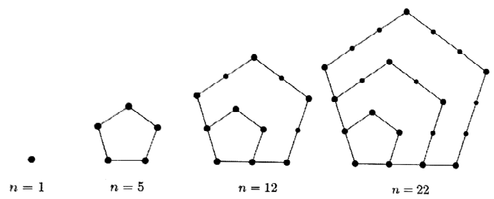
20. The number of pairs (a, b) of real numbers such that the zeros of the polynomials $(6x^2-24x-4a)$ and (x^3+ax^2+bx-8) are all non-negative real numbers is _____.

21. The number of pairs of positive integers (m, n) such that m x n = gcd (m, n) + lcm (m, n) is _______.

22. Two spheres of radii 3 cm and 4 cm, whose centres are 5 cm apart cut each other. The volume common to both the spheres is given by $\frac{a}{b}\pi$ cm^3 where $\frac{a}{b}$ is the fraction with the lowest numerator and denominator, then the value of $\frac{a}{b}$ is ______.

23. The number of points with integer coordinates on the line x + y = 1 which are located inside the circle with centre (0, 0) and radius 3 is ______.

24. Look at the pattern shown in the figure Pentagonal Numbers.

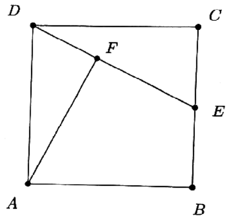


25. The number of natural numbers $n \le 2013$ for which $2^n - 1$ is divisible by 7 is _____.

26. A fruit basket contains 4 bananas, 5 oranges and 6 apples. The number of ways a person can make a selection of the fruits from among the fruits in the basket is ______.

27. a, b, c are positive integers such that a + b + c = 2013. Given that a! b! c! = $m10^n$ where m, n are integers and m is not divisible by 10, the smallest value of n is _____.

28. In the figure (i) below, ABCD is a square. E is the midpoint of CB. AF is drawn perpendicular to DE. If the side of the square is 2013 cm the length of FB is ______ cm.



29. The number of primes p for which (p + 2) and $p^2 + 2p - 8$ are both primes is ______.

30. f(x) is a linear function. f(0) = –5 and f(f(0)) = –15. The number of values of α for which the solutions to the inequality f(x) f(α – x) > 0 form an interval of length 2 is ____.

PART - A

1. When x, y, z are real, the minimum value of $2x^2 + 2y^2 + 5z^2 - 2xy - 4xz - 4yz - 4x - 2x + 15$ is

(A) 18 (B) 25 (C) 10 (D) 15

2. If f(x) is a real values function and (for any x real ≠ 0) $3f(x) - 2f\left(\frac{f}{x}\right) = x$, the value of f(4) is

(A) $\frac{5}{2}$ (B) $\frac{7}{2}$ (C) $\frac{1}{2}$ (D) 0

3. The number of natural numbers n for which (3n – 4), (4n – 5), (5n – 3) are all prime number is

(A) 0 (B) 1 (C) 2 (D) infinite

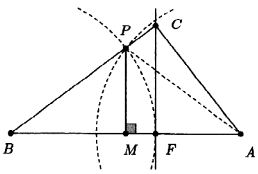
4. Five points on a circle are numbered 1, 2, 3, 4 and 5. A frog jumps in a clockwise direction from one point to another round the circle. If it is on an odd numbered point, it jumps one point, and if it is on an even–numbed point, it moves two points. The frog begins on point 5. After 2004 jumps it will be on point

(A) 1 (B) 2 (C) 3 (D) 4

5. The number of solutions of the equation $(\log_{10}x)^2 = \log_{10}(100x)$ is

(A) 0 (B) 1 (C) 2 (D) 4

6. ABC is a right angled triangle with hypotenuse AB and CF is the altitude. The circle with center at B and passing through F and another circle of the same radius with center at A insersect on the side BC. Then the ratio $\frac{BF}{BC}$ is equal to



(A) $\frac{1}{2}$ (B) $\frac{1}{\sqrt{2}}$ (C) $\frac{1}{\sqrt[3]{2}}$ (D) $\frac{1}{2\sqrt{2}}$

7. The number of natural numbers n, for which n! + 5 is a perfect cube is

(A) 0 (B) 1 (C) 2 (D) 5

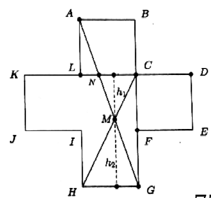
8. Two sides of a triangle are 8 cm and 18 cm and the bisector of the angle formed by them is of length $\frac{60}{13}$ cm. The length of the third side (in cm) is

(A) 22 (B) 23 (C) 24 (D) 25

9. A number of workers in their tea time went to a tea shop and took tea and snacks. At the end they decided to split the bill evenly among them. If each contributes Rs. 16, they found that they were 4 Rs. short, while if each contributes Rs. 19, they had enough to pay the bill, 15% for the tip and Rs. 2 left over. The number of workers is

(A) 10 (B) 12 (C) 15 (D) 11

10. Consider the figure ABCDEFGHIJKL as shown. Each side is of length 4 and the angle between any two consecutive sides is a right angle. AG and CH cut at M. The area of the quadrilateral ABCM is



(A) $\frac{88}{5}$ (B) $\frac{44}{3}$ (C) $\frac{77}{9}$ (D) $\frac{62}{3}$

11. The number of pairs of natural numbers (x, y) which satisfy $\frac{5}{x}+\frac{6}{y}=1$ is

(A) 5 (B) 30 (C) 11 (D) 8

12. The sides of a triangle are 15, 20 and 25. The length of the shortest altitude is

(A) 6 (B) 12 (C) 10 (D) 13

13. a, b, c are reals such that a – 7b + 8c = 4 and 8a + 4b – c = 7. The value of $a^2 - b^2 + c^2$ is

(A) 0 (B) 12 (C) 8 (D) 1

14. n is a five digit number. If q and r are respectively the quotient and remainder when n is divided by 100, the number of n for which (q + r) is divisible by 11 is

(A) 8181 (B) 8180 (C) 8182 (D) 9000

15. A sphere is inscribed in a cube that has a surface area of 24 cm^2. A second cube is then inscribed within the sphere. The surface area of the inner cube in square centimeters is

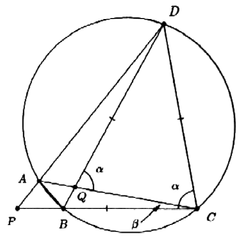
(A) 3 (B) 8 (C) 6 (D) 9

PART - B

16. The smallest multiple of 15 such that the result contains only 0 or 8 is ______.

17. Vishwa is walking up a stair that has 10 steps and with each stride the goes up either one step of two steps. The number of different ways Vishwa can go up the stars is ______.

18. The quadrilateral ABCD is inscribed in a circle. The diagonals AC and BD cut at Q. DA produced and CB produced cut at P. If CD = CP = DQ, then ∠DAC = ______.



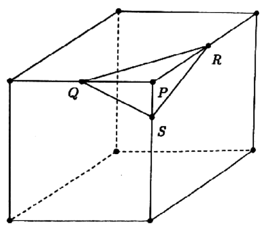
19. The sum of the first 100 terms of a arithmetic progression is –1, and sum of the 2^{nd}, 4^{th}, 6^{th}, 8^{th}, ... and the 100^{th} terms is 1. Then the sum of the squares of the first 100 terms of the A.P. is ______.

20. The number of pairs of positive integers (m, n) such that m, n have no factors greater that 1 and $\frac{m}{n}+\frac{14n}{9m}$ is an integer is ______.

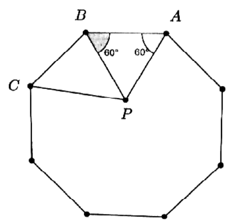
21. The number of two digit numbers that increase by 75% when their digits are reversed is _____.

22. ABCD is a rectangle. A is (14, –32), B is (2014, 168) and D is (10, y) for some integer y. The area of the rectangle is ______.

23. P is the vertex of cuboid. Q, R, S are points on the edges shown. If PQ = 4 cm, PR = 4 cm and PS = 2 cm and the area of triangle QRS is $\sqrt{K}$ cm^2 then K = ______.



24. ABCDEFGH is a regular octagon. ABP is an equilateral triangle with P inside the octagon. Then measure of 2∠APC = _______.



25. The number of numbers from 12 to 12345 inclusive having digits which are consecutive and in increasing order reading from left to right is _______.

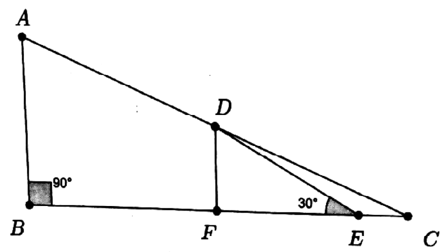
26. The number of integer pairs (m, n) such that $15m^2 - 7n^2 = 9$ is _______.

27. A positive integer n is a multiple of 7. If $\sqrt{n}$ lies between 15 and 16, the number of possible values of n is ______.

28. The number 27000001 has exactly 4 prime factors. The sum of these prime factors is _____.

29. a is an integer such that $\frac{a}{23!} = 1 + \frac{1}{2} + \frac{1}{3} + ... + \frac{1}{23}$. The remainder when a is divided by 13 is ____.

30. ABC is a right triangle as shown. D is the midpoint of AC. E is a point on BC such that $\angle DEB = 30^0$. DE = K · AB where K is a number. Then the value of K is ______.

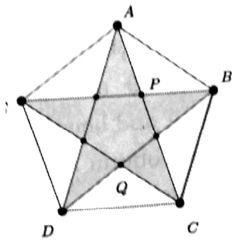


PART - A

1. P is a point on AL, the altitude of the triangle ABC through A. If $\angle PBA = 20^0$, $\angle PBC = 40^0$ and $\angle PCB = 30^0$, then $\angle PCA$ equals

(A) 20^0 (B) 10^0 (C) 15^0 (D) 18^0

2. ABCDE is a regular pentagon. The area of the star shaped region ACEBDA is 1 square cm. AC and BE meet at P and BD and CE meet at Q as shown in the figure. The area of APQD in square cms is



(A) $\frac{1}{4}$ (B) $\frac{1}{3}$ (C) $\frac{1}{2}$ (D) $\frac{3}{4}$

3. The number of integer pairs (x, y) which satisfy the equation $x^3 = y^3 + 2y^2 + 1$ is

(A) 0 (B) 1 (C) 2 (D) 3

4. Let $f(x) = x^4 + x^3 + x^2 + x + 1$. The remainder when $f(x^5)$ is divided by f(x) is

(A) 5 (B) 6 (C) 7 (D) 4

5. The number of positive integers a, b, c such that $a^2 + b^2 + c^2 = a^2b^2$ is

(A) 1 (B) 2 (C) 5 (D) none of these

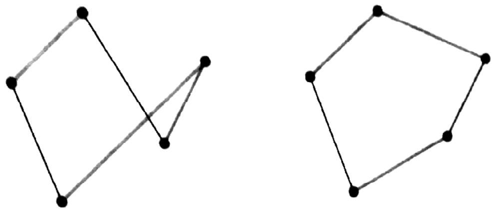
6. The number of triples (x, y, z) of real numbers such that $3x^2 + y^2 + z^2 = 2x(y + z)$ is

(A) 1 (B) 2 (C) 3 (D) 5

7. The number of real roots of the equation $x^4 - 4x = 1$ is

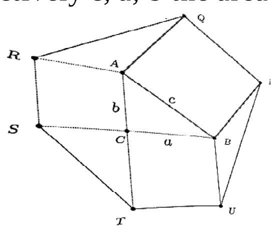
(A) 1 (B) 2 (C) 0 (D) 4

8. Given a set of r points in the plane so that no three are collinear, by a closed polygon we mean the polygon obtained by connecting them by r line segments as shown in the examples below in Figure 2 (here r = 5). There are 10 points on a plane no three of which are collinear. The numbe of 5 sided closed polygons whose vertices are among these 10 points is



(A) 6048 (B) 1507 (C) 3024 (D) 10000

9. In the adjacent figure ABC is a right angled triangle. Squares are described externally on its sides and the outer vertices of these squares are joined as shown. If the lengths of the sides AB, BC, CA are respectively c, a, b the area of the hexagon PQRSTU is



(A) $2(a^2 + ab + b^2)$ (B) $a^2 + ab + b^2$

(C) $3(a^2 + ab + b^2)$ (D) $\frac{1}{2}(a^2 + ab + b^2)$

10. ABCD is a square. From B, D lines are drawn to meet at P inside the square such that $\angle ADP = 25^0$ and $\angle ABP = 20^0$. Then $\angle BPC$ is

(A) 70^0 (B) 80^0 (C) 60^0 (D) 50^0

11. If p, q are positive odd integers such that (1 + 3 + 5 + ... + p) + (1 + 3 + 5 + ... + q) = 1 + 3 ..... + 19 then p + q is

(A) a prime number (B) divisible by 13

(C) odd number (D) none of these

12. The number $2^{20} - 1$ is divisible by

(A) 11 and 41 (B) 11 and 21 (C) 41 and 61 (D) 11 and 61

13. Five points O, A, B, C, D are taken in order on a straight line such that OA = a, OB = b, OC = c and OD = d. P is a point on the line between B and C. If AP : PD = BP : BC, then OP is

(A) $\frac{ac - bd}{a - b + c - d}$ (B) $\frac{ac + bd}{a - b + c - d}$ (C) $\frac{ad - bc}{a - b + c - d}$ (D) none of these

14. The side AB of an equilateral triangle AB is produced to D such that BD = 2AB. The point F is the foot of the perpendicular from D on CB produced. $\angle FAC$ =

(A) 70^0 (B) 75^0 (C) 80^0 (D) 90^0

15. For the simultaneous equations $x^2 + 2xy + y^2 - x - y = 6$, $x - 2y = 3$

(A) there is a solution (x, y) such that both x, y are irrational

(B) there are two sets of solutions (x, y) such that x, y are integers

(C) sum of all solutions is 1

(D) product of all solutions is $\frac{5}{3}$

PART - B

16. The number of right angled triangles with integer side lengths and such that the product of the lengths of the legs (non–hypotenuse sides) equals three times the perimeter of the triangle is _______.

17. $\alpha, \beta, \gamma, \delta$ are the roots of the equation $x^4 - ax^3 + ax^2 + bx + c = 0$, where a, b, c are real numbers. The smallest possible value of $\alpha^2 + \beta^2 + \gamma^2 + \delta^2$ is ______.

18. Two circles with centers P and Q and radii 3 and 4 respectively touch each other externally. AB, CD are direct common tangents touching the smaller circle at A, C and the bigger circle at B, D. The area of the concave hexagon APCDQB is _______.

19. If a, b are the lengths of unequal diagonals of a regular heptagon (regular polygon with 7 sides) with side c, then $\frac{1}{a} + \frac{1}{b}$ in terms of c is _______.

20. The number of real roots of the equation $\frac{\log_{10}(\sqrt{x+1}+1)}{\log_{10}\sqrt[3]{x-40}} = 3$ is ______.

21. x, y, z are non zero real numbers such that $x^2 + y^2 + z^2 = 1$

$x\left(\frac{1}{y}+\frac{1}{z}\right)+y\left(\frac{1}{z}+\frac{1}{x}\right)+z\left(\frac{1}{x}+\frac{1}{y}\right)+3=0$ The number of possible values of x + y + z is ______.

22. The minimum value of integer n such that among any n integers we can always find three integers whose sum is divisible by 3 is ______.

23. The number of integers n for which $n^4 - 51n^2 + 50$ is negative is _______.

24. In a triangle ABC, the lengths of the sides are consecutive integers and the median drawn from A is perpendicular to the bisector of angle B. The largest side of the triangle has length _____.

25. a, b, c, d, e are real numbers such that

$$a + 4b + 9c + 16d + 25e = 1$$
$$4a + 9b + 16c + 25d + 36e = 8$$
$$9a + 16b + 25c + 36d + 49e = 23$$

The value of a + b + c + d + e is _______.

26. In a triangle ABC, the altitude, angle bisector and the median from C divide the angle C into four equal angles. The measure of the least angle of the triangle is ______.

27. AB is a chord of a circle with center O. AB is produced to C such that BC = OA. CO is produced to E. The value of $\frac{\angle AOE}{\angle ACE}$ is ______.

28. The number of two digit numbers that are less than the sum of the squares of their digits by 11 and exceed twice the product of their digits by 5 is _______.

29. ABD is a circle whose centre is C. The circle circumscribing ABC cuts DA or DA produced at E. Then the triangle BDE is a _______ triangle.

30. The number of 4-digit numbers N such that
    (a) no digit of N is 9
    (b) N is the square of an integer
    (c) when each digit of N is increased by 1, the resulting number is also square of an integer

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Note :

1. Fill in the response sheet with your Name, Class and the institution through which you appear in the specified places.
2. Diagrams are only visual aids; they are NOT drawn to scale.
3. You are free to do rough work on separate sheets.
4. Duration of the test : 2 pm to 4 pm – 2 hours.

PART - A

Note :

- Only one of the choices A, B, C, D is correct for each question. Shade the alphabet of your choice in the response sheet. If you have any doubt in the method of answering, seek the guidance of the supervisor.
- For each correct response you get 1 mark. For each incorrect response you lose $\frac{1}{2}$ mark.

1. The equations $5x^2 - 10x\cos\alpha + 7\cos\alpha + 6 = 0$ has two identical roots. If α is one of the angles of a parallelogram with sum of the lengths of two adjacent sides equal to 6, the maximum area of the parallelogram is

(A) 32/5 (B) 34/5 (C) 36/5 (D) 18/5

2. If f(x) is a polynomial of degree three with leading coefficient 1 such that f(1) = 1, f(2) = 4, f(3) = 9, then the value of f(4) is

(A) 16 (B) 22 (C) 36 (D) None of these

3. The three roots of the equation $3x^3 + px^2 + qx - 4$ are the side length, inradius and the circumradius of an equilateral triangle. Then the value of 2p + q is

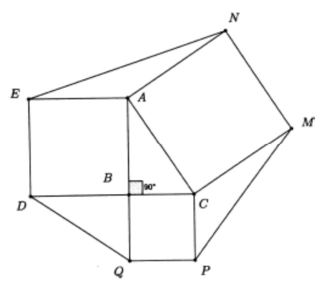
(A) –10 (B) 10 (C) –12 (D) 12

4. Given $a = -\sqrt{99} + \sqrt{999} + \sqrt{9999}$, $b = \sqrt{99} - \sqrt{999} + \sqrt{9999}$, $c = \sqrt{99} + \sqrt{999} - \sqrt{9999}$ the value of $\frac{a^4}{(a-b)(a-c)} + \frac{b^4}{(b-c)(b-a)} + \frac{c^4}{(c-a)(c-b)}$

(A) 22194 (B) $\sqrt{99} + \sqrt{999} + \sqrt{9999}$

(C) 22190 (D) $\sqrt{99 \times 999 \times 9999}$

5. In the figure, ABC is a right angled triangle with $\angle B = 90^0$, AB = 8 cm and BC = 6 cms. Squares ANMC, AEDB, BQPC are described as shown, on the sides AC, AB, BC respectively. Among PM, NE and DQ.



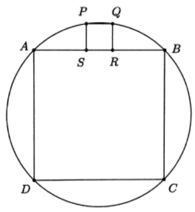
(A) Two are of integral length and the length of the other is irrational

(B) Two are of irrational length and the length of the other is integral

(C) All have irrational lengths

(D) All have integral length

6. In the adjoining figure, ABCD and PQRS are squares. If the length of the side of the bigger square is a, the length of the side of the smaller square is



(A) $\frac{a}{3}$ (B) $\frac{a}{4}$ (C) $\frac{a}{5}$ (D) $\frac{a}{6}$

7. Twelve people sit around a circular table. Each observes that his age (viewed as an integer) is the average of the ages of his left and right neighbours. Which of the following could be the sum of their ages ?

(A) 224 (B) 226 (C) 227 (D) 228

8. A rectangular billiard table has vertices at (0, 0), (12, 0), (0, 10), (12, 10). There are pockets only in the four corners. A ball is hit from the corner (0, 0) along the line $y = x$ and bounces off several walls before eventually entering a pocket. The number of walls that the ball bounces off before entering a pocket is

(A) 7 (B) 8 (C) 9 (D) 11

9. In a quadrilateral ABCD, we have AB = 8, BC = 5, CD = 17 and DA = 10. The diagonals AC and BD meet at E. If BE = ED = 1 : 2, the area of the quadrilateral ABCD is

(A) 70 (B) 60

(C) 50 (D) Not uniquely determined

10. Let S(n) denote the sum of the digits of the integer n when n is written in the usual decimal form. For example, S(123) = 6. If S(n) = 1274, what is a possible value of S(n + 1) ?

(A) 1239 (B) 1266 (C) 1275 (D) 1284

11. The polynomial $g(x) = x^3 + ax^2 + bx + c$ has three distinct roots and each root of $g(x) = 0$ is also a root of $f(x) = x^4 + x^3 + bx^2 + 100x + c$. What is f(1) ?

(A) –5005 (B) –6006 (C) –7007 (D) –8008

12. The number N = 1234567891011....41424344 is the 79 digit number obtained by writing the numbers from 1 to 44 in order. What is the remainder when N is divided by 45 ?

(A) 1 (B) 9 (C) 18 (D) 27

13. Define the sequence Fn recursively as follows : $F_0 = 1$, $F_1 = 1$ and for $n \geq 2$, F_n is the remainder of $F_{n-1} + F_{n-2}$ when divided by 3. What is the value of $\sum_{k=2017}^{2024} F_k$?

(A) 6 (B) 7 (C) 8 (D) 9

14. P, Q, R, S have integer coordinates and are distinct points on the circle $x^2 + y^2 = 25$. The distance PQ and RS are irrational numbers. What is the largest possible value of PQ/RS ?

(A) 3 (B) $5\sqrt{2}$ (C) 7 (D) $3\sqrt{5}$

15. In how many ways can 1000 be written as a sum of 2s and 3s, ignoring order

(500 x 2 + 0 x 3 and 50 x 2 + 300 x 3 are two of the ways) ?

(A) 500 (B) 499 (C) 167 (D) 166

PART - B

Note :

- Write the correct answer in the space provided in the response sheet.
- For each correct response you get 1 mark. For each incorrect response you lose $\frac{1}{4}$ mark.

16. The six digit number 789ABC consists of six distinct digits and is divisible by 7, 8 and 9. The three digit number ABC is ________.

17. m, n are relatively prime positive integers such that $\frac{m}{n} = \frac{2(\sqrt{2}+\sqrt{10})}{5\sqrt{3+\sqrt{5}}}$, then m + n equals ____.

18. ABC is a triangle with AB = 17 units. F is the mid point of AB and CF = 8 units. The maximum possible area of the triangle ABC is ______.

19. Given that a + b + c = 5 and $1 \le a, b, c \le 2$ the minimum value of $\frac{1}{a+b} + \frac{1}{b+c}$ is _______.

20. The number of ways we can insert + 's between the digits of 111111111111111111 (eighteen 1's) so that the result will be a multiple of 90 is ________.

21. The largest positive integer less than 2017 that has exactly three proper factors (a proper factor is a factor other than the number of itself; for example, 11 has only one proper factor) is _______.

22. Consider the sequence 1, 3, 4, 7, 11, 18, 29, ... in which each term from the third term onwards is the sum of the two previous terms. Of the first 100 terms of this sequence the number of terms that are multiples of 5 is ________.

23. The non negative, distinct integers a, b, c, d, e form an arithmetic progression. If the sum of the numbers is 440, the maximum possible value of e is _______.

24. Let f be defined for all positive integers as follows : $f(n) = \begin{cases} n^2+1 & \text{if } n \text{ is odd} \\ \frac{n}{2} & \text{if } n \text{ is even} \end{cases}$ The number of integers n such that $1 \le n \le 100$ for which $f(f(...f(n))) = 1$ (where f is applied some number of times) is _______.

25. M and N are three digit numbers with no digit equal to zero. If we rearrange the digits of M all the numbers obtained are less than M . N has the same digits as M but in a different order. If we rearrange the digits of N, the largest number we obtain is M. If M + N = 1233, M equals _________.

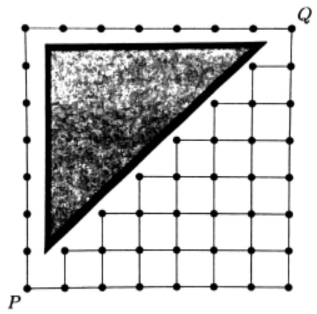
26. ABCDEF is a hexagon inscribed in a circle of radius R. If AB = CD = EF = 2 and BC = DE = FA = 10, the radius R is ________.

27. a, b, c are real numbers satisfying the following equations :
$$\log_2(abc - 3 + \log_5 a) = 5$$
$$\log_3(abc - 3 + \log_5 b) = 4$$
$$\log_4(abc - 3 + \log_5 c) = 4$$
The value of $|\log_5 a| + |\log_5 b| + |\log_5 c|$ is _______.

28. In a triangle ABC, I is the incenter. The internal angle bisector of ∠C meets AB at F and the circum circle of triangle ABC at Z. If FI = 2, ZF = 3 and IC = m/n where m, n are relatively prime positive integers, the value of m + n is _________.

29. The largest integer such that $n^3 + 4n^2 - 15n - 18$ is a perfect cube is _____.

30. The grid below shows a network of roads and pond (there are 8 horizontal lines and 8 vertical lines in the figure). You can move only horizontally or vertically from one grid point to an adjacent grid point. You do not know swimming and hence need to avoid going through the triangular pond at the top left corner of the grid. The number of shortest paths between P and Q is ________.



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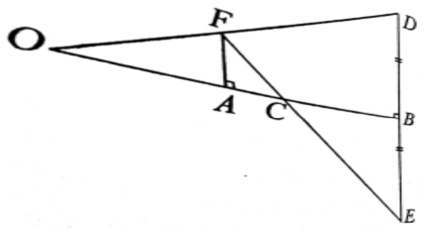
1. Find all integers $n \geq 1$ such that $\frac{n^3+3}{n^2+7}$ is an integer.

2. A point is chosen on each side of a unit square. The four points form the sides of a quadrilateral with sides of lengths a, b, c, d. Show that $2 \leq a^2+b^2+c^2+d^2 \leq 4$
$2\sqrt{2} \leq a+b+c+d \leq 4$

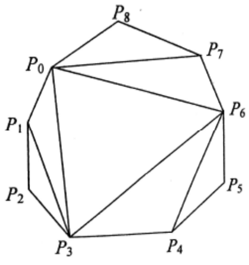
3. ABCD is a convex quadrilateral inscribed in a circle Σ. Assume that A, B and Σ are fixed and C, D are variable points, so that the length of the segment CD remains constant. Points X and Y are on the rays AC and BC respectively such that AX = AD and BY = BD. Prove that the distance between X and Y remains constant.

4. In the adjoining figure, OB is the perpendicular bisector of DE. A is a point on OB; AF is perpendicular to OB and EF intersects OB at C. Show that OC is the harmonic mean between OA and OB. i.e., $OC = \frac{2 \cdot OA \cdot OB}{OA + OB}$.



5. Find all integral values of x, y, z, w given that $x! + y! = 2^z 3^w$.

6. A convex polygon of nine vertices $P_0, P_1, P_2, \ldots, P_8$ is given along with six diagonals as shown in diagram. We see that 7 triangles $P_0P_1P_3$, $P_0P_3P_6$, $P_0P_6P_7$, $P_0P_7P_8$, $P_1P_2P_3$, $P_3P_4P_6$ and $P_4P_5P_6$ are created. These triangles are to be numbered $\Delta_1, \Delta_2, \Delta_3, \ldots, \Delta_7$ so that P_i is a vertex of Δ_i. In how many ways can this be done ? Justify your answer.



7. Let f(x) be a linear function such that f(0) = –5 and f(f(0)) = –15. Find all values of m for which the solutions of the inequality f(x) f(m – x) > 0 form an interval of length 2.

8. In how many ways can you select two disjoint subsets from a set having n elements ?

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1. Let O and I be respectively the circumcentre and incentre of a triangle ABC. Given $C = 30^0$. Let E and D be points respectively on AC and BC such that AE = AB = BD. Show that DE = IO and DE and IO are perpendicular to each other.

2. Let N be a 2n digit number with digits $d_1, d_2, d_3, \ldots, d_{2n}$ from left to right (i.e.) $N = d_1d_2 \ldots d_{2n}$ where $d_i \neq 0$, i = 1, 2, 3, ..., 2n. Find the number of such N so that the sum $d_1 \times d_2 + d_2 + d_3 \times d_4 + d_5 \times d_6 + \ldots + d_{2n-1} \times d_{2n}$ is even.

3. $P_1, P_2, \ldots, P_n$ be n points on a circle (in order) dividing the circumference into n equal arcs. Find a permutation $Q_1, Q_2, \ldots, Q_n$ of these points such that the sum of the lengths of the path $Q_1Q_2 + Q_2Q_3 +, \ldots + Q_{n-1}Q_n$ is maximum.

4. AA^1 is the median of the triangle ABC. BE, CF are the altitudes of the triangle ABC, cutting at the orthocenter H. The line joining E, F meets BC produced at Q. Show that H is also the orthocenter of the triangle AA^1Q.

5. Let $\frac{p}{q} = 1 + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{256}$ where (p, q) = 1. Prove that p is divisible by 257^2 (257 is a prime).

6. a, b, c are real numbers such that abc + a + c = b and ac ≠ 1. Find the greatest value of the expression $\left(\frac{2}{a^2+1} - \frac{2}{b^2+1} + \frac{3}{c^2+1}\right)$.

7. $x_1, x_2, \ldots, x_n$ $(n \geq 2)$ are reals satisfying $\frac{1}{x_1+2011} + \frac{1}{x_2+2011} + \ldots + \frac{1}{x_n+2011} = \frac{1}{2011}$. Show that $\frac{\sqrt[n]{x_1x_2\ldots x_n}}{(n-1)} \geq 2011$.

8. ABC is a scalene triangle. Equilateral triangles ABC_1, BCA_1, CAB_1 are drawn outwards of the triangle ABC.

   Prove that

   (a) AA_1, BB_1, CC_1 are concurrent (at a point K say)

   (b) $AA_1 = KA + KB + KC$.

RAMANUJAN CONTEST - FINAL - INTER LEVEL

1. Find all the pairs (x, y) where x, y are integers satisfying $(2x - 1)^3 + 16 = y^4$.
2. I is the incentre of the isosceles triangle ABC in which AB = AC. Let Σ be a circle which touches AB at E and AC at F and touches the circumcircle of triangle ABC internally. Prove that I lies on EF.
3. A function $f : Q \to Q$, where is the set of rational numbers, satisfies the conditions
   (a) f(1) = 2.
   (b) f(xy) + f(x + y) = f(x) . f(y) + 1 for all x, y ∈ Q. Determinate all such functions f, with proof.
4. Let B be a point on the circle Σ_1 and A be a point on the tangent at B to Σ_1 (B ≠ A). Let C be a point not on Σ_1 such that AC meets Σ_1 in two distinct points. Let Σ_2 be a circle touching AC at C and Σ_1 and D on the same side of AC as B. Prove that the circumcentre of triangle BCD lies on the circumcircle of triangle ABC.
5. Find all positive integers x, y, z such that $8^x + 15^y = 17^x$.
6. A Pythagorean triangle is a right triangle in which all the three sides are of integer lengths. Let a, b be the legs of a Pythagorean triangle, and h be altitude to the hypotenuse. Determine all such a triangles for which $\frac{1}{a} + \frac{1}{b} + \frac{1}{h} = 1$.
7. Let f(n) be a function defined on the non-negative integer n. Given
   (a) f(0) = f(1) = 0
   (b) f(2) = 1
   (c) for n > 2, f(n) gives the smallest positive integer which does not divide n.
8. In a circle C with centre O and radius r, let C_1 and C_2 be two circles with centres O_1, O_2 and radii r_1 and r_2 respectively be situated such that each circle C_1 and C_2 is internally tangent to C at A_1 and A_2 respectively and such that C_1 and C_2 are externally tangent to each other at A. Prove that the three lines OA, O_1A_2 and O_2A_1 are concurrent.

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1. ABC is a triangle in which AB > AC > BC. D is a point on the minor arc BC of the circumcircle of the triangle ABC. O is the circumcentre. E and F are the intersection points of the line AD with the perpendiculars from O to AB and AC respectively. P is the point of intersection of BE and CF. If PB = PC + PO, find the angle A of the triangle ABC.

2. a) For the positive integer n define $f(n) = 1^n + 2^{n-1} + 3^{n-2} + ... + (n-2)^3 + (n-1)^2 + n^1$.

   What is the minimum value of $\frac{f(n+1)}{f(n)}$?

   b) ABC is an isosceles triangle in which AB = AC. The bisector of ∠B meets AC at D and it is given that BC = BD + AD. Find ∠A of the triangle ABC.

3. Let n be a positive integer and S_n be the set of all positive integer divisors of n (including 1 and itself). Prove that at most half of the elements of S_n have their units digit equal to 3.

4. a) Let A be a set of 8 elements. Find the maximum number of 3 – element subsets of A, such that the intersection of any two of them is not a 2 element set.

   b) a, b, c, d, are all positive reals and $\frac{1}{1+a^4}+\frac{1}{1+b^4}+\frac{1}{1+c^4}+\frac{1}{1+d^4}=1$. Prove that $abcd \geq 3$.

5. In a plane there are two similar, convex quadrilaterals ABCD and $AB_1C_1D_1$ such that C, D are inside $AB_1C_1D_1$ and B is outside $AB_1C_1D_1$. Prove that if the lines BB_1, CC_1, DD_1 are concurrent, then ABCD is cyclic. Is the converse true ?

6. Prove that if the integer n is not divisible by 5, then the polynomial $f(x) = x^5 - x + n$ cannot be factored as the product of two non-constant polynomials with integer coefficients.

7. a) One may perform the following two operations on a positive integer.

   (i) Multiply it by any positive integer.

   (ii) Delete zeros in its decimal representation.

   b) Show that 1! + 2! + 3! + ... + 2013! can not be written as n^k for any integer n and integer $k \geq 2$.

8. a) $\lfloor x \rfloor$ denotes the floor function (the greatest integer function). Let r be a real number for which $\left\lfloor r+\frac{19}{100}\right\rfloor+\left\lfloor r+\frac{20}{100}\right\rfloor+\left\lfloor r+\frac{21}{100}\right\rfloor+...+\left\lfloor r+\frac{91}{100}\right\rfloor=546$. Solve the equation $x+\lfloor 100r \rfloor = 2013$.

   b) For all distinct positive integers m and n prove $(2013)^{2^n} + 2^{2^n}$ is relatively prime to $(2013)^{2^m} + 2^{2^m}$.

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1. a) If x and y are positive reals such that $x^{2014} + y^{2014} = 1$ prove that
$$\left(\sum_{k=1}^{1007} \frac{1+x^{2k}}{1+x^{4k}}\right)\left(\sum_{k=1}^{1007} \frac{1+y^{2k}}{1+y^{4k}}\right) < \frac{1}{(1-x)(1-y)}.$$

   b) The angles of a triangle are in arithmetic progression. The altitudes of this triangle are also in arithmetic progression. Show that the triangle is equilateral.

2. ABC is an acute angled triangle in which the three sides are unequal. L, M, N are respectively the midpoints of the sides BC, CA and AB. The perpendicular bisectors of the sides AB and AC intersect AL at D and E respectively. BD and CE cut at F inside the triangle. Show that A, M, F, N are concyclic.

3. S is a finite set of positive integers such that for any two distinct numbers of S, a and b, the number $\dfrac{a+b}{\gcd(a,b)}$ is also a member of S. Determine all such sets S.

4. a) Find all real number triples (x, y, z) which satisfy
$$3(x^2 + y^2 + z^2) = 1$$
$$x^2y^2 + y^2z^2 + z^2x^2 = xyz(x+y+z)^3$$

   b) Show that all the integers 1 to 16 can be written in a line, but not in a circle, so that the sum of any two adjacent numbers is a perfect square.

5. A certain number is the product of three different prime factors, the sum of whose squares is 2331. There are 7560 numbers (including unity) which are less than the number and prime to it. The sum of all its divisors (including unity and the number) is 10560. Find the number.

6. A rectangular parallelepiped is given, such that its intersection with a plane is a regular hexagon. Prove that the rectangular parallelepiped is a cube.

7. A triangle ABC is given. The midpoints of the sides AC and AB are B_1 and C_1 respectively. The incenter of the ΔABC is I. The lines B_1I and C_1I meet the sides AB, AC at C_2 and B_2 respectively. If the areas of ΔABC and ΔAB_2C_2 are equal, find the measure of the angle BAC.

8. If (a + b) and (a – b) are relatively prime integers for some natural numbes a, b. Find the greatest common divisor of $2a + (1 - 2a)(a^2 - b^2)$ and $2a(a^2 + 2a - b^2)(a^2 - b^2)$.

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1. A, B, C are three points on a circle. The distance of C from the tangents at A and B to the circle are a and b respectively. If the distance of C from the chord AB is c, show that c is the geometric mean of a and b.

2. Find all integer solutions to the equation $x^3 + (x + 4)^2 = y^2$.

3. Two right angled triangles are such that the incircle of one triangle is equal in size to the circum circle of the other. If Δ_1 is the area of the first triangle and Δ_2, the area of the second triangle, show that $\frac{\Delta_1}{\Delta_2} \geq 3 + 2\sqrt{2}$.

4. (a) Find the maximum value k for which one can choose k integers from 1, 2, ..., 2n so that none of the chosen integers is divisible by any other chosen integer.

   (b) F(x) is a polynomial of degree 2016 such that all the coefficients are non negative and none exceed F(0). Show that the coefficient of x^{2017} in $(F(x))^2$ is at most $\frac{F(1)^2}{2}$.

5. (a) $n \geq 3$ and $a_1, a_2, \ldots, a_n$ are different positive integers. Given that, except the first and the last, each one is a harmonic means of its immediate neighbors. Show that none of the given integers is less than $n - 1$.

   (b) Show that the shortest side of a cyclic quadrilateral with circumradius 1 is at most $\sqrt{2}$.

6. C_1, C_2, C_3 are circles with radii 1, 2, 3 respectively, touching each other as shown. Two circles can be drawn touching all these three circles. Find the radii of these two circles.
