Regional Mathematical Olympiad - 2024

Time: 3 hours **November 3, 2024**

Instructions:

- **Calculators (in any form) and protractors are not allowed.**
- **Rulers and compasses are allowed.**
- **All questions carry equal marks. Maximum marks: 102.**
- **No marks will be awarded for stating an answer without justification.**
- **Answer all the questions.**
- **Answer to each question should start on a new page. Clearly indicate the question number.**

1. Let $n > 1$ be a positive integer. Call a rearrangement $a_1, a_2, \ldots, a_n$ of $1, 2, \ldots, n$ *nice* if for every $k = 2, 3, \ldots, n$, we have that $a_1 + a_2 + \cdots + a_k$ is **not** divisible by k.

   (a) If $n > 1$ is odd, prove that there is no *nice* rearrangement of $1, 2, \ldots, n$.

   (b) If n is even, find a *nice* rearrangement of $1, 2, \ldots, n$.

2. For a positive integer n, let $R(n)$ be the sum of the remainders when n is divided by $1, 2, \ldots, n$. For example, $R(4) = 0+0+1+0 = 1$, $R(7) = 0+1+1+3+2+1+0 = 8$. Find all positive integers n such that $R(n) = n - 1$.

3. Let ABC be an acute triangle with $AB = AC$. Let D be the point on BC such that AD is perpendicular to BC. Let O, H, G be the circumcentre, orthocentre and centroid of triangle ABC respectively. Suppose that $2 \cdot OD = 23 \cdot HD$. Prove that G lies on the incircle of triangle ABC.

4. Let a_1, a_2, a_3, a_4 be real numbers such that $a_1^2 + a_2^2 + a_3^2 + a_4^2 = 1$. Show that there exist i, j with $1 \leq i < j \leq 4$, such that $(a_i - a_j)^2 \leq \frac{1}{5}$.

5. Let $ABCD$ be a cyclic quadrilateral such that AB is parallel to CD. Let O be the circumcentre of $ABCD$, and L be the point on AD such that OL is perpendicular to AD. Prove that
$$OB \cdot (AB + CD) = OL \cdot (AC + BD).$$

6. Let $n \geq 2$ be a positive integer. Call a sequence $a_1, a_2, \cdots, a_k$ of integers an *n-chain* if $1 = a_1 < a_2 < \cdots < a_k = n$, and a_i divides a_{i+1} for all $i, 1 \leq i \leq k-1$. Let $f(n)$ be the number of *n-chains* where $n \geq 2$. For example, $f(4) = 2$ corresponding to the 4-chains $\{1, 4\}$ and $\{1, 2, 4\}$.

   Prove that $f(2^m \cdot 3) = 2^{m-1}(m+2)$ for every positive integer m.

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